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Local discontinuous Galerkin methods with novel basis for fractional diffusion equations with non-smooth solutions

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Abstract In this paper, we develop a novel local discontinuous Galerkin (LDG) methods for fractional diffusion equations with non-smooth solutions. We consider such problems, for which the solutions are not smooth at boundary, and therefore the traditional LDG methods with piecewise polynomial solutions suffer accuracy degeneracy. The novel LDG methods utilize a solution information enriched basis, and simulate the problem on a paired special mesh, and achieve optimal order of accuracy. We analyze the L^2-stability and optimal error estimate in L^2-norm. Finally, numerical examples are presented for validating the theoretical conclusions.

Keywords local discontinuous Galerkin methods · fractional diffusion equations · non-smooth solutions · novel basis · optimal order of accuracy

Mathematics Subject Classification (2010) MSC 65M60

1 Introduction

Much attention has been attracted to fractional differential equations (FDEs) in recent years because of their strong ability to model certain processes which can not be adequately described by usual partial differential equations [12,17]. These models are found in a wide range of applications such as porous flows, biological processes, and transport in fusion plasma to name a few [7].

The wide application of FDEs interests people to design high order numerical method for them. However, there are two main difficulties when applying traditional numerical method, such as Spectral methods [1,10], Local Discontinuous Galerkin (LDG) method [3,11] on FDEs (i) fractional derivatives are non-local

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operators; (ii) fractional derivatives involve singular kernel/weight functions, and the solutions of FDEs are usually weakly singular near the boundaries and at initial time.

Recently, some high-order methods have been developed for FDEs, like finite difference schemes [2, 26], spectral methods [15, 16] and local discontinuous Galerkin method [6–8] by assuming that the solution is sufficiently smooth. However, the assumption that the solution of FDEs is sufficiently smooth, does not hold in general for FDEs.

For example, in [7], local discontinuous Galerkin methods were developed for fractional diffusion equations, and demonstrated to achieve optimal order of accuracy both theoretically and computationally for smooth enough underlying solutions. However, the order degeneracy is observed when applied to problems with non-smooth solutions. Consider solving a non-smooth problem with a solution $u(\cdot\,,t) \in H^{\alpha}$, the LDG method [7] using finite element space V^k of piecewise polynomials with degree up to k, as defined in (2.11), will achieve an error with a theoretical order $\min\{k+1, \alpha\}$, and the numerical order is observed to be like $\min\{k+1, \alpha+0.5\}$[1].

In [22], Local discontinuous Galerkin method with non-polynomial basis was proposed, and applied to simulate multiscale problems [20, 25], Schrödinger equation in a resonant tunneling diode [21], convection dominated problems [18] and time-harmonic problems [9]. The idea is to choose suitable novel basis based for the problem to achieve optimal order of accuracy. For example, the boundary layer problems usually have a large slope of the solution near the boundary, which can be better approximated by exponential functions rather than polynomials. The problem with high oscillatory solution is better approximated by trigonometric functions. In this work, we use polynomials with fractional orders to approximate the solution near weak singularity to achieve better order resolution than regular polynomials.

In this paper, we develop and analyze novel LDG methods with non-polynomial basis to simulate fractional diffusion equations with non-smooth solutions. The goal is to design proper basis for non-smooth problems, so that optimal order of accuracy is achieved. Consider the equations in the form

$$\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + f(x,t), \quad x \in [a,b] \tag{1.1}$$

with appropriate initial condition and Dirichlet boundary conditions. Here, the order of derivative $\beta \in (1,2)$, the constant $d > 0$ is the generalized diffusion coefficient, and $f(x,t)$ is a source term. For example, consider

$$\frac{\partial u(x,t)}{\partial t} = \frac{2}{3\Gamma(1.5)}\frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} - e^{-t}(x^{1.5}+1), \quad x \in (0,1), \tag{1.2}$$

with initial condition $u(x,0) = x^{1.5}$ and boundary conditions $u(0,t) = 0$ and $u(1,t) = e^{-t}$. Here $\Gamma(x)$ is the classic Gamma function. The exact solution is $u(x,t) = e^{-t}x^{1.5}$, and has a weak singularity at the left boundary of the domain and $u(\cdot,t) \in H^{1.5}(0,1)$. In Example 9, we solve this problem with LDG method with space V^2 on uniform cells, and only get second order convergence, see third

[1] We would like to thank Jan Hesthaven for helpful discussions.

column of Table 9; and in Example 10, we also try to solve fractional differential equations (6.5) with a solution $u(x,t) = e^{-t}x^{\frac{4}{3}}$ using LDG with spaces V^1 and V^2 on uniform cells, and the orders of accuracy are both about 1.8, see third columns in Table 11 and 12. It is clear that using the regular piecewise polynomials to simulate the problem suffers from losing order of accuracy, and this inspire us to try novel basis space for such problems.

In this work, we always consider the singularity at the left boundary $x = a$ for simplicity. Note that the same methodology is similar for the singularity being at a different location, thanks to the local nature of the discontinuous Galerkin methods.

The rest of the paper is organized as follows. In Section 2, we list some preliminaries, including fractional calculus in Section 2.1, the novel approximation space in Section 2.2.1, and the design of mesh in Section 2.2.2. The detail of the novel LDG methods are described in Section 3. The stability analysis is shown in Section 4, and the accuracy analysis is in Section 5. The numerical observations in Section 6 validate our theory. Some conclusion and future work are in Section 7.

2 Preliminaries

In this section, we list some preliminaries. We first recall some definitions and lemmas in fractional calculus in Section 2.1, which will be used in analysis in later sections, and then introduce the approximation space and the special mesh in Section 2.2.

2.1 Fractional Calculus

We consider functions whose support is in (a, b). The formal definition of the fractional integral emerges as a natural generalization of multiple integration and defined as follows.

Definition 1 (Fractional integral) Let $\alpha \in \mathbb{R}^+$. The left-side and right-side Riemann-Liouville fractional integrals ${}_aD_x^{-\alpha}v(x)$ and ${}_xD_b^{-\alpha}v(x)$ can be defined as

$$ {}_aD_x^{-\alpha}v(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\xi)^{\alpha-1}v(\xi)d\xi, \quad x > a, \tag{2.1}$$

and

$$ {}_xD_b^{-\alpha}v(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (x-\xi)^{\alpha-1}v(\xi)d\xi, \quad x < b, \tag{2.2}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}$ and a can be $-\infty$ and b can be $+\infty$.

There are several definitions of fractional derivatives, including Riemann-Liouville derivative, Caputo derivative, Grünwald-Letnikov derivative and so forth. The Riemann-Liouville derivative is recovered by first performing integration followed by classic differentiation as follows:

Definition 2 (Riemann-Liouville derivative)

$$ {}_aD_x^{\alpha}v(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{\mathrm{d}x^n}\int_a^x (x-\xi)^{n-\alpha-1}v(\xi)\mathrm{d}\xi, \quad x > a, \quad \alpha \in [n-1, n) \tag{2.3} $$

The Caputo derivative is achieved by reversing the order of integration and differentiation, and of the form

Definition 3 (Caputo derivative)

$$ {}_a^C D_x^{\alpha}v(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-\xi)^{n-\alpha-1}\frac{d^n v(\xi)}{\mathrm{d}\xi^n}\mathrm{d}\xi, \quad x > a, \quad \alpha \in [n-1, n) \tag{2.4} $$

These two definitions are connected, but different. For example, ${}_0D_x^{\alpha}x^{\beta} = {}_0^C D_x^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}$ for $\beta > \alpha$, however, ${}_0D_x^{\alpha}1 = \frac{1}{\Gamma(1-\alpha)}x^{-\alpha}$, ${}_0^C D_x^{\alpha}1 = 0$. The connection between the two definitions can be shown in the following Lemma [13].

Lemma 1 *For Riemann-Liouville derivative (2.3) and Caputo derivative (2.4),*

$$ {}_aD_x^{\alpha}v(x) = {}_a^C D_x^{\alpha}v(x), \qquad \text{if } v^{(k)}(a) = 0, \quad k = 0, 1, \ldots, n-1 \tag{2.5} $$

where n is the smallest integer greater than or equal to α, and $v(x)$ is sufficiently smooth.

Note that the solution defined in our work satisfies this theorem.

Lemma 2 *(see [24], Proposition 1) The left and right Riemann-Liouville fractional integrate operators are adjoint with respect to the $L^2(a, b)$ inner product, i.e., for all $\alpha > 0$*

$$ ({}_aD_x^{-\alpha}v,\, v)_{L^2} = ({}_aD_x^{-\frac{\alpha}{2}}v,\, {}_xD_b^{-\frac{\alpha}{2}}v)_{L^2} = (v,\, {}_xD_b^{-\alpha}v)_{L^2} \tag{2.6} $$

Lemma 3 *(see [7], Lemma 2.6)*

$$ ({}_{-\infty}D_x^{-\alpha}v,\, {}_xD_{\infty}^{-\alpha}v)_{L^2} = \cos(\alpha\pi)\|{}_{-\infty}D_x^{-\alpha}v\|_{L^2}^2 \tag{2.7} $$

Lemma 4 *(see [7] Theorem 2.8) For $-\alpha_2 < -\alpha_1 < 0$,*

$$ \|{}_{-\infty}D_x^{-\alpha_2}v\|_{L^2} \le C\|{}_{-\infty}D_x^{-\alpha_1}v\|_{L^2}. \tag{2.8} $$

for some constant $C > 0$.

2.2 The approximation space and the special mesh

In this section, we introduce the novel approximation space and the mesh that is specially designed, based on the solution information.

Consider that the solution to problem (1.1) has an weak singularity at the left end of the domain, and is of form

$$ w(x)(x-a)^{\beta} \tag{2.9} $$

with some unknown but smooth function $w(x)$. We would like to get optimal approximation of the solution with a novel approximation space and a properly chosen mesh.

Let

$$a = x_0 < x_1 < \cdots < x_n = \hat{x} < \cdots < x_{M-1} < x_M = b \tag{2.10}$$

be a division of domain $[a, b]$, where $\hat{x} \in (a, b)$ is a grid node x_n $(0 < n < M)$ and also a dividing point that needs to be determined later. We define the mesh $\mathcal{T} := \{I_j = (x_{j-1}, x_j),\ j = 1, \ldots, M\}$, and the cell length $h_j := |I_j| = x_j - x_{j-1}$.

The standard piecewise polynomial functions in space V^k, defined as

$$V^k = \left\{ v : v|_{I_j} \in P^k(x),\ \text{for any cell } I_j \right\}, \tag{2.11}$$

work well while approximating the solution away from the weak singularity $x = a$. Therefore, we adopt such functions as basis functions in cells I_j with $n < j \le M$. Here, $\hat{x} = x_n$ needs to be big enough for this statement to hold true.

Near the end $x = a$, we adopt the mapped polynomial functions $P^k((x-a)^\gamma)$ as basis functions to approximate the solution, where $\gamma = \frac{1}{\gamma'}$ with γ' being the smallest positive integer makes $\gamma'\beta \in \mathbb{N}$. Since $\beta \in (1, 2)$, it is clear that $\gamma \in (0, 1)$. With such non-polynomial finite element spaces, the approximation for such solutions of FDEs is expected to obtain better accuracy.

2.2.1 Approximation space

We now define the finite element space V_γ^k [2] for both trial functions and test functions:

$$V_\gamma^k = \left\{ v : v\big|_{I_j} \in P^k((x-a)^\gamma),\ \text{if } x_j \le \hat{x};\ v\big|_{I_j} \in P^k(x),\ \text{otherwise} \right\}. \tag{2.12}$$

In order to overcome the lack of regularity of the solution, we choose $P^k((x-a)^\gamma)$ as basis for a subdomain near the weak singularity. Therefore, in the domain near starting point the exact solution, being considered as a function of the mapped variable $y = (x-a)^\gamma$, is a regular function .

For a piecewise function $v(x) \in V_\gamma^k$ on mesh $\mathcal{T}$, the two values of $v(x)$ read from two sides of any node x_j might be different. The one-sided limits at the nodes are defined by

$$v^\pm(x_j) = v(x_j^\pm) := \lim_{x \to x_j^\pm} v(x). \tag{2.13}$$

2.2.2 Special mesh

By using the above non-polynomial basis $P^k((x-a)^\gamma)$ in the cells $I_j = (x_{j-1}, x_j)$, with $x_j \lesssim \hat{x}$, near the dividing point $\hat{x}$, the mass matrix becomes ill-conditioned with high condition number, which is worsen for finer mesh. This fact will influence the accuracy in numerical simulations.

In summary, we need to design the mesh $\mathcal{T}$ (2.10) so that the followings are satisfied.

1. The dividing point $\hat{x}$ is chosen properly such that the standard polynomial basis well approximates the solution on uniform cells inside $[\hat{x}, b]$.

[2] Any function $v \in V_\gamma^k$ is a piecewise fractional polynomial, that is, on each cell I_j, v is of form $a_{k,j}(x-a)^{k\gamma} + a_{k-1,j}(x-a)^{(k-1)\gamma} + \cdots + a_{1,j}(x-a)^\gamma + a_{0,j}$ with real coefficients $a_{k,j}, a_{k-1,j}, \ldots, a_{1,j}$ and $a_{0,j}$.

2. The grid nodes $\{x_1, x_2, \cdots, x_{n-1}\}$ are chosen to make the non-polynomial basis in the cells on left of dividing point $\hat{x}$ not ill-conditioned, especially while mesh being refined.

Graded mesh is widely used in singular problems [5, 23]. In order to avoid the ill-conditioned mass matrix, we adopt graded mesh for the cells with irregular basis, that is, for $I_j = (x_{j-1}, x_j)$ with $x_j \leq \hat{x}$. Under the mapping $y = (x-a)^\gamma$, for $x \in [a, \hat{x}]$, one gets a set of mapped nodes $y_j = (x_j - a)^\gamma$ for $j = 0, 1, \ldots, n$ and mapped cells $\tilde{I}_j = (y_{j-1}, y_j)$. The graded mesh $\{x_j\}_{j=0}^n$ requires that $\tilde{h}_j = y_j - y_{j-1} = (x_j - a)^\gamma - (x_{j-1} - a)^\gamma$, $j = 1, \ldots, n$, are all the same.

There are freedoms on choosing the dividing point $\hat{x}$ and the mesh sizes in two subdomains, i.e. $(a, \hat{x})$ and $(\hat{x}, b)$. Here is a demonstration on mesh design. For the simplicity of accuracy analysis, we tend to choose the mesh sizes on the two subdomains "agree", that is,

$$\tilde{h} = h, \tag{2.14}$$

where $\tilde{h} = \tilde{h}_j = (x_j - a)^\gamma - (x_{j-1} - a)^\gamma$ for all $1 \leq j \leq n$, and $h = h_j = x_j - x_{j-1}$ for all $n < j \leq M$.

To achieve this result, we assume that $h = \frac{b-a}{N}$ with some $N \in \mathbb{N}_{>0}$. In the subdomain $[\hat{x}, b]$, there are $\frac{b-\hat{x}}{b-a}N$ uniform cells with cell size h. To determine the mesh nodes $\{x_j\}_{j=1}^{n-1}$ in subdomain $(a, \hat{x})$, one needs to set the location of $\hat{x}$ and number of nodes n. To make sure (2.14) holds true, it is required that

$$n \cdot h = (\hat{x} - a)^\gamma. \tag{2.15}$$

As a result, the number of cells in total is

$$M = \frac{b-\hat{x}}{b-a}N + \frac{(\hat{x}-a)^\gamma}{b-a}N. \tag{2.16}$$

The nodes are, for $0 \leq j \leq n$,

$$\begin{aligned} y_j &= jh = \frac{b-a}{N}j, \\ x_j &= a + y_j^{1/\gamma} = a + \left(\frac{b-a}{N}j\right)^{1/\gamma} = a + \left(\frac{b-a}{N}j\right)^{\gamma'}; \end{aligned} \tag{2.17}$$

for $n < j \leq M$,

$$x_j = b - (M-j)h = b - (M-j)\frac{b-a}{N}. \tag{2.18}$$

There is still one freedom to choose the parameter pair $\{\hat{x}, n\}$ to satisfy the mesh condition (2.15). We choose the ones those are easy to define and refine. Here are some examples.

Example 1 Consider a problem defined in $[0, 1]$. For $\beta = \frac{3}{2}$, $\gamma' = 2$, and thus $\gamma = \frac{1}{2}$. Therefore, the mapped polynomials are $y^k = x^{\frac{k}{2}}$, $k = 0, 1, 2, \ldots$. For optimal mesh size, we choose $\hat{x} = \frac{1}{4}$, and thus there are $n = \frac{(\frac{1}{4})^{\frac{1}{2}}}{1/N} = \frac{N}{2}$ graded cells and $\frac{3}{4}N$ uniform cells in the subdomain $[0, \frac{1}{4}]$ and $[\frac{1}{4}, 1]$, respectively. So $M = \frac{5}{4}N$.

Example 2 Consider a problem defined in $[0,1]$. For $\beta = \frac{4}{3}$ or $\frac{5}{3}$, $\gamma' = 3$, and thus $\gamma = \frac{1}{3}$. For optimal mesh size, we choose $\hat{x} = \frac{1}{8}$, and thus there are $n = \frac{(\frac{1}{8})^{\frac{1}{3}}}{1/N} = \frac{N}{2}$ graded cells and $\frac{7}{8}N$ uniform cells in the subdomain $[0, \frac{1}{8}]$ and $[\frac{1}{8}, 1]$, respectively. So $M = \frac{11}{8}N$.

Note that the mesh defined in Example 2 needs N be dividable by 8. The following is an alternative way to design a mesh for the novel basis.

Example 3 Instead we divide the subdomain $[\frac{1}{8}, 1]$ into cells with a smaller cell length $\frac{h}{4} = \frac{1}{4N}$. Then there are $M = \frac{N}{2} + \frac{7N}{2} = 4N$ cells in total, and N just needs to be dividable by 2, which allows larger cell size in subdomain $[a, \hat{x}]$, compared with Example 2. For this mesh, the mesh size is noted by a pair $\{\frac{1}{N}, \frac{1}{4N}\}$.

3 The novel local discontinuous Galerkin methods for the fractional diffusion equation

In this section, we will describe the algorithm formulation. The idea of local discontinuous Galerkin methods for time dependent problems with higher derivatives, such as the convection diffusion equation, is to introduce auxiliary variables and rewrite the equation into a first order system, and then apply the discontinuous Galerkin method on the system. We refer to [4, 14, 19] for detailed discussions. The LDG schemes for fractional diffusion equations (1.1) are designed in a similar fashion.

Since we consider the solution u of form (2.9) with a weak singularity at $x = a$, it is assumed that $u(a,t) \equiv 0$, and thus the solution satisfies (2.5) with $\alpha = \beta - 1$ in Lemma 1. As a result, we have the following:

$$\frac{\partial^\beta u(x,t)}{\partial x^\beta} = \frac{\partial}{\partial x}\, {}_aD_x^{\beta-2}\, \frac{\partial}{\partial x} u(x,t). \tag{3.1}$$

Therefore, the equation (1.1) can be rewritten into the following system

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} - \sqrt{d}\frac{\partial q(x,t)}{\partial x} = f(x,t) && \text{in } \Omega_T, \\ &q - {}_a D_x^{\beta-2} p(x,t) = 0 && \text{in } \Omega_T, \\ &p - \sqrt{d}\frac{\partial u(x,t)}{\partial x} = 0 && \text{in } \Omega_T, \end{aligned} \tag{3.2}$$

with initial condition $u(x,0) = u_0(x)$ on Ω, and Dirichlet boundary conditions $u(a,t) = 0$ and $u(b,t) = g(t)$ on $(0,T)$. Here $\Omega_T = (a,b) \times (0,T)$, $\Omega = (a,b)$ and p and q are two auxiliary variables.

The novel finite element space is V_γ^k as defined in (2.12), where γ is determined by β as described in Section 2.2. The semi-discrete LDG scheme to solve system (3.2) is defined as follows. Find $u_h, q_h, p_h \in V_\gamma^k$ such that, for all test functions

$v, w, z \in V_\gamma^k$ and all $j = 1, 2, \ldots, M$, we have

$$\begin{aligned}
&\left(\frac{\partial u_h(x,t)}{\partial t},\, v(x)\right)_{I_j} + \sqrt{d}\left(q_h(x,t),\, \frac{\partial v(x)}{\partial x}\right)_{I_j} - \sqrt{d}\hat{q}(x,t)v(x)\Big|_{x_{j-1}^+}^{x_j^-} = (f(x,t),\, v(x))_{I_j},\\
&\qquad (q_h(x,t),\, w(x))_{I_j} - ({}_aD_x^{\beta-2}p_h(x,t),\, w(x))_{I_j} = 0,\\
&(p_h(x,t),\, z(x))_{I_j} + \sqrt{d}\left(u_h(x,t),\, \frac{\partial z(x)}{\partial x}\right)_{I_j} - \sqrt{d}\hat{u}(x,t)z(x)\Big|_{x_{j-1}^+}^{x_j^-} = 0,
\end{aligned} \tag{3.3}$$

where the notation $(a(x,t),\, b(x))_{I_j} := \int_{x_{j-1}}^{x_j} a(x,t)\, b(x)\, dx$. The function u_h is initialized as:

$$\begin{aligned}
(u_h(x(y),0),\, v\circ x(y))_{\tilde{I}_j} &= (u_0\circ x(y),\, v\circ x(y))_{\tilde{I}_j}\,, &&\text{if } j \le n;\\
(u_h(x,0)\,,\, v(x))_{I_j} &= (u_0(x),\, v(x))_{I_j}\,, &&\text{if } j > n.
\end{aligned} \tag{3.4}$$

The "hat" terms $\hat{q}(x,t)$ and $\hat{u}(x,t)$ are numerical fluxes [3], and single valued functions defined at the cell interfaces, which usually depends on the values read from both sides. To determine the choice on such numerical fluxes, one needs to consider not only locality, but also consistency, stability, and accuracy of the resulted scheme. Here, we use the so-called "alternating fluxes", which is a popular and attractive choice and defined as

$$\hat{q}(x_j,t) = q_h^+(x_j,t), \quad \hat{u}(x_j,t) = u_h^-(x_j,t); \tag{3.5}$$

or

$$\hat{q}(x_j,t) = q_h^-(x_j,t), \quad \hat{u}(x_j,t) = u_h^+(x_j,t) \tag{3.6}$$

at any interior cell interfaces; at the domain boundaries,

$$\hat{u}(a,t) = 0, \quad \hat{u}(b,t) = g(t), \tag{3.7}$$

and

$$\hat{q}(a,t) = q_h^+(a,t), \quad \hat{q}(b,t) = q_h^-(b,t), \tag{3.8}$$

which reflect the Dirichlet boundary conditions.

4 Stability analysis

In this section, we discuss about the stability.

The scheme (3.3) can be expressed as: find $u_h,\, p_h,\, q_h \in V_\gamma^k$ such that for all $v,\, w,\, z \in V_\gamma^k$ and all $j = 1, 2, \ldots, M$, the following holds

$$\mathcal{B}_j(u_h, p_h, q_h; v, w, z) = \mathcal{L}_j(v, w, z), \tag{4.1}$$

where the bilinear form $\mathcal{B}_j$ is defined as

$$\begin{aligned}
\mathcal{B}_j(u_h, p_h, q_h; v, w, z) := &\left(\frac{\partial u_h(x,t)}{\partial t},\, v(x)\right)_{I_j} + \sqrt{d}\left(q_h(x,t),\, \frac{\partial v(x)}{\partial x}\right)_{I_j}\\
&+ (q_h(x,t),\, w(x))_{I_j} - ({}_aD_x^{\beta-2}p_h(x,t),\, w(x))_{I_j}\\
&+ (p_h(x,t),\, z(x))_{I_j} + \sqrt{d}\left(u_h(x,t),\, \frac{\partial z(x)}{\partial x}\right)_{I_j}\\
&- \sqrt{d}\,\hat{q}(x,t)v(x)\Big|_{x_{j-1}^+}^{x_j^-} - \sqrt{d}\,\hat{u}(x,t)z(x)\Big|_{x_{j-1}^+}^{x_j^-},
\end{aligned} \tag{4.2}$$

and the linear form $\mathcal{L}_j$ is defined as

$$\mathcal{L}_j(v,w,z) = (f(x,t),\, v(x))_{I_j}. \tag{4.3}$$

By summing over all cells, we have

$$\mathcal{B}(u_h,p_h,q_h;v,w,z) = \mathcal{L}(v,w,z), \tag{4.4}$$

with the discrete bilinear form

$$\begin{aligned}\mathcal{B}(u_h,p_h,q_h;v,w,z) := &\left(\frac{\partial u_h(x,t)}{\partial t},\, v(x)\right)_{L^2} - ({}_aD_x^{\beta-2}p_h(x,t),\, w(x))_{L^2}\\ &+ (q_h(x,t),\, w(x))_{L^2} + (p_h(x,t),\, z(x))_{L^2}\\ &+ \sqrt{d}\left(q_h(x,t),\, \frac{\partial v(x)}{\partial x}\right)_{L^2} + \sqrt{d}\left(u_h(x,t),\, \frac{\partial z(x)}{\partial x}\right)_{L^2}\\ &- \sqrt{d}\sum_{j=1}^{M}\left(\hat{q}(x,t)v(x) + \hat{u}(x,t)z(x)\right)\Big|_{x_{j-1}^{+}}^{x_j^{-}},\end{aligned} \tag{4.5}$$

and the discrete linear form

$$\mathcal{L}(v,w,z) = (f(x,t),\, v(x))_{L^2}. \tag{4.6}$$

Consider $\widetilde{u_h}$, $\widetilde{p_h}$, $\widetilde{q_h}$ as the perturbed solutions of u_h, p_h, q_h, which means $\widetilde{u_h}$, $\widetilde{p_h}$, $\widetilde{q_h}$ satisfy (4.1) with perturbed initial condition $\widetilde{u_h}(x,0) = \widetilde{u_0}(x)$. Denote the differences

$$e_{u_h} := \widetilde{u_h} - u_h, \quad e_{p_h} := \widetilde{p_h} - p_h, \quad e_{q_h} := \widetilde{q_h} - q_h. \tag{4.7}$$

The stability of scheme (3.3) is concluded as follows.

Theorem 1 *(L^2 stability) The scheme (3.3) is L^2 stable, and the solutions satisfies, for all $t \in [0,T]$,*

$$\|e_{u_h}(\cdot,t)\|_{L^2}^2 + 2\cos((\beta/2-1)\pi)\int_0^t \|{}_aD_x^{\beta/2-1}e_{p_h}(\cdot,t)\|_{L^2}^2\, dt = \|e_{u_h}(\cdot,0)\|_{L^2}^2. \tag{4.8}$$

Proof It is clear that solutions $\widetilde{u_h}$, $\widetilde{p_h}$, $\widetilde{q_h}$ satisfy the perturbation equation

$$\mathcal{B}(\widetilde{u_h},\widetilde{p_h},\widetilde{q_h};v,w,z) = \mathcal{L}(v,w,z) \tag{4.9}$$

for all v, w, $z \in V_\gamma^k$.

Subtracting (4.4) from (4.9) gives

$$\mathcal{B}(e_{u_h},e_{p_h},e_{q_h};v,w,z) = 0, \tag{4.10}$$

for any v, w, $z \in V_\gamma^k$, which allows taking $v = e_{u_h}$, $w = -e_{p_h}$, $z = e_{q_h}$. Therefore,

$$\mathcal{B}(e_{u_h},e_{p_h},e_{q_h};e_{u_h},-e_{p_h},e_{q_h}) = 0. \tag{4.11}$$

From the definition of $\mathcal{B}$ in (4.5),

$$\begin{aligned}
&\mathcal{B}(e_{u_h}, e_{p_h}, e_{q_h}; e_{u_h}, -e_{p_h}, e_{q_h}) \\
&= \left(\frac{\partial e_{u_h}}{\partial t}, e_{u_h}\right)_{L^2} + ({}_aD_x^{\beta-2} e_{p_h}, e_{p_h})_{L^2} + (e_{q_h}, -e_{p_h})_{L^2} + (e_{p_h}, e_{q_h})_{L^2} \\
&\quad + \sqrt{d}\left[\left(e_{q_h}, \frac{\partial e_{u_h}}{\partial x}\right)_{L^2} + \left(e_{u_h}, \frac{\partial e_{q_h}}{\partial x}\right)_{L^2} - \sum_{j=1}^{M} (\hat{e}_{q_h} e_{u_h} + \hat{e}_{u_h} e_{q_h})\Big|_{x_{j-1}^+}^{x_j^-}\right] \\
&= \frac{1}{2}\frac{\partial}{\partial t}(e_{u_h}, e_{u_h})_{L^2} + \cos((\beta/2-1)\pi)\|{}_aD_x^{\beta/2-1} e_{p_h}\|_{L^2}^2 \\
&\quad + \sqrt{d}\sum_{j=1}^{M}\int_{x_{j-1}}^{x_j} \frac{\partial}{\partial x}(e_{u_h} e_{q_h})\, dx + (\hat{e}_{q_h} e_{u_h} + \hat{e}_{u_h} e_{q_h})\Big|_{x_{j-1}^+}^{x_j^-},
\end{aligned} \tag{4.12}$$

where in the last equation, we use the fact from Lemma 2 and Lemma 3 that

$$({}_aD_x^{\beta-2} e_{p_h}, e_{p_h})_{L^2} = ({}_aD_x^{\frac{\beta}{2}-1} e_{p_h}, {}_xD_b^{\frac{\beta}{2}-1} e_{p_h})_{L^2} = \cos((\frac{\beta}{2}-1)\pi)\|{}_aD_x^{\frac{\beta}{2}-1} e_{p_h}\|_{L^2}^2. \tag{4.13}$$

The last term in (4.12) depends on the way to choose numerical fluxes. Take the choice (3.5) as an example, which makes it equal to

$$\begin{aligned}
&\sqrt{d}\sum_{j=1}^{M}[e_{u_h} e_{q_h} - \hat{e}_{q_h} e_{u_h} - \hat{e}_{u_h} e_{q_h}]\Big|_{x_{j-1}^+}^{x_j^-} \\
=&\sqrt{d}\sum_{j=2}^{M-1}\left[e_{u_h}^- e_{q_h}^- - e_{q_h}^+ e_{u_h}^- - e_{u_h}^- e_{q_h}^-\right]\Big|_{x_j} - \left[e_{u_h}^+ e_{q_h}^+ - e_{q_h}^+ e_{u_h}^+ - e_{u_h}^- e_{q_h}^+\right]\Big|_{x_{j-1}} \\
&- \sqrt{d}\left[e_{u_h}^+ e_{q_h}^+ - e_{q_h}^+ e_{u_h}^+ - e_{u_h}^- e_{q_h}^+\right]\Big|_{x_{M-1}} + \sqrt{d}\left[e_{u_h}^- e_{q_h}^- - e_{q_h}^+ e_{u_h}^- - e_{u_h}^- e_{q_h}^-\right]\Big|_{x_1} \\
=&\sqrt{d}\sum_{j=2}^{M-1}\left(-e_{u_h}^- e_{q_h}^+\Big|_{x_j} + e_{u_h}^- e_{q_h}^+\Big|_{x_{j-1}}\right) + \sqrt{d}\left(e_{u_h}^- e_{q_h}^+\Big|_{x_{M-1}} - e_{u_h}^- e_{q_h}^+\Big|_{x_1}\right) = 0,
\end{aligned} \tag{4.14}$$

where, in the first equation, we use the fact that both e_{u_h} and $\hat{e}_{u_h}$ vanish at the two boundaries $x = a$ and $x = b$. Note that the other choice of numerical flux (3.6) also makes this interface term equal to zero.

Combing (4.12) and (4.14), yields

$$\frac{1}{2}\frac{\partial}{\partial t}\|e_{u_h}(\cdot,t)\|_{L^2}^2 + \cos((\beta/2-1)\pi)\|{}_aD_x^{\beta/2-1} e_{p_h}\|_{L^2}^2 = 0. \tag{4.15}$$

The result follows as one integrates (4.15) over time, and uses the simple fact that

$$\frac{1}{2}\int_0^t \frac{\partial}{\partial s}\|e_{u_h}(\cdot,s)\|_{L^2}^2\, ds = \frac{1}{2}\|e_{u_h}(\cdot,t)\|_{L^2}^2 - \frac{1}{2}\|e_{u_h}(\cdot,0)\|_{L^2}^2. \tag{4.16}$$

□

Remark 1 Note that $\beta \in (1,2)$, and thus the equation (4.8) implies that

$$\|e_{u_h}(\cdot,t)\|_{L^2} \le \|e_{u_h}(\cdot,0)\|_{L^2}. \tag{4.17}$$

5 Accuracy analysis

In this section, we conduct error analysis for the proposed scheme. We start with approximation error of non-polynomial basis functions in (2.12) on mesh $\mathcal{T}$ (2.10) in Section 5.1. We also give error estimates for some L^2 projections in this section. The error estimate of scheme is summarized in Theorem 2, which is proved in Section 5.2.

5.1 Approximation theory

In this section, we show the proposed non-polynomial finite element space V_γ^k (2.12) on mesh $\mathcal{T}$ (2.10) approximates the solution of form (2.9) with optimal order of accuracy. The approximation results are listed here, and the proofs are provided in the appendix sections.

First, we recall the following approximation theory of the traditional piecewise polynomial functions on a uniform mesh.

Lemma 5 *For a L^2 projection $\mathcal{P}_h$ into a finite element space V^k of piecewise polynomials with degree up to k on a uniform mesh with size h, and a given smooth function ω, the projection $\mathcal{P}_h\omega$ is a unique function in this space which satisfies*

$$\|\mathcal{P}_h\omega - \omega\|_{L^2} \leq Ch^{k+1}. \tag{5.1}$$

Note that C is a generic constant depending on function ω and its derivatives but independent of h throughout this paper.

However, the lemma requires a target function with high regularity. The performance of traditional basis to approximate function with low regularity can be seen in Section 6.1. From the numerical results, it is easy to find that for a function in H^α, the numerical order of approximation is about $\min\{k+1, \alpha+0.5\}$.

We consider two types of projections into space V_γ^k (2.12): the first one is a standard L^2 projection, and the other one is tailored with respect to the mapping $y = (x-a)^\gamma$, which is used in initialization of the scheme.

Lemma 6 *(Approximation theorem for standard L^2 projection) For a L^2 projection $\mathcal{S}$ into space V_γ^k (2.12), and a function ω of form (2.9) with one weak regularity at the left boundary of domain $[a,b]$, the projection $\mathcal{S}\omega$ is a unique function in V_γ^k which satisfies*

$$\|\mathcal{S}\omega - \omega\|_{L^2} \leq Ch^{k+1}. \tag{5.2}$$

The optimal convergence rate of approximation in Lemma 6 is validated in numerical results of Example 5, Example 6 and Example 7.

Next we consider a projection $\tilde{\mathcal{P}}_h$ onto space V_γ^k: on uniform cell I_j with $j > n$, the projection $\tilde{\mathcal{P}}_h$ is conducted in the L^2 sense; on graded cell I_j with $j \leq n$, the L^2 projection is with respect to the y-domain under mapping $y = (x-a)^\gamma$.

Lemma 7 *(Approximation theorem for the tailored L^2 projection) For the L^2 projection $\tilde{\mathcal{P}}_h$ defined above, and a function ω of form (2.9), the projection $\tilde{\mathcal{P}}_h\omega$ is a unique function in V_γ^k which satisfies*

$$\|\tilde{\mathcal{P}}_h\omega - \omega\|_{L^2} \leq Ch^{k+1}. \tag{5.3}$$

Remark 2 Note that in the proofs, the cell length h in both (A.4) and (B.2) is of the mapped cells $\tilde{I}_j$, denoted as $\tilde{h}$ previously. Using graded mesh in the subdomain $\in [a, \hat{x}]$ gives $\tilde{h} = h$. This is the motivation of adopting graded mesh around the weak singularity. See more details about notations in Section 2.2.2.

Remark 3 Lemma 7 guarantees that the initialization (3.4) is of optimal order of accuracy.

In order to provide error estimate for the scheme, we need to define some projections and provide estimations of projection errors. Let $\mathcal{P}^{\pm}$ be two projection operators onto the novel finite element space V_γ^k. For intervals $I_j = (x_{j-1}, x_j)$, $j = 1, 2, \ldots, M$, $\mathcal{P}^{\pm}$ are defined to satisfies the $k+1$ conditions, respectively:

$$\begin{aligned} &(\mathcal{P}^{\pm}u - u,\, v)_{I_j} = 0, \qquad \forall v \in V_\gamma^{k-1}, \\ &\mathcal{P}^{-}u(x_j) = u^{-}(x_j), \qquad \mathcal{P}^{+}u(x_{j-1}) = u^{+}(x_{j-1}). \end{aligned} \tag{5.4}$$

And $\mathcal{S}'$ is a standard L^2-projection onto space V_γ^{k-1}, and defined as follows:

$$(\mathcal{S}'u - u,\, v)_{I_j} = 0, \qquad \forall v \in V_\gamma^{k-1}. \tag{5.5}$$

For projections $\mathcal{P}^{\pm}$, we list the error estimations in the following theorem.

Lemma 8 *(Projection error estimate) For projections $\mathcal{P}^{\pm}$ defined in (5.4),*

$$\|\mathcal{P}^{\pm}\omega - \omega\|_{L^2} \le Ch^{k+1}. \tag{5.6}$$

5.2 Error estimates

Here we estimate the error of the novel LDG scheme for fractional diffusion equation with non-smooth solutions.

Theorem 2 *The error for the scheme (3.3) with flux (3.5) or (3.6) and (3.7)-(3.8) satisfies*

$$\|u - u_h\|_{L^2} \le Ch^{k+1}. \tag{5.7}$$

Proof We denote

$$e_u = u(x,t) - u_h(x,t), \quad e_p = p(x,t) - p_h(x,t), \quad e_q = q(x,t) - q_h(x,t). \tag{5.8}$$

Since the solution u, p and q to system (3.2) satisfies

$$\mathcal{B}(u, p, q; v, w, z) = \mathcal{L}(v, w, z). \tag{5.9}$$

Subtracting (4.4) from (5.9) gives

$$\mathcal{B}(e_u, e_p, e_q; v, w, z) = 0. \tag{5.10}$$

Take

$$v = \mathcal{P}^{\pm}u - u_h, \quad w = p_h - \mathcal{S}p, \quad z = \mathcal{P}^{\mp}q - q_h \tag{5.11}$$

and let

$$\xi_u = \mathcal{P}^{\pm}u - u, \quad \xi_p = p - \mathcal{S}p, \quad \xi_q = \mathcal{P}^{\mp}q - q. \tag{5.12}$$

Note the order of projections is related to the choice of numerical fluxes. Thanks to the definitions (5.4) and (5.5), after rearranging terms, we obtain

$$\mathcal{B}(v,-w,z;v,w,z)=\mathcal{B}(\xi_u,-\xi_p,\xi_q;v,w,z). \tag{5.13}$$

Similar to the proof for stability, we have

$$\begin{aligned}\mathcal{B}(v,-w,z;v,w,z)&=\frac{1}{2}\frac{\partial}{\partial t}\|v(\cdot,t)\|_{L^2}^2+\left({}_aD_x^{\beta-2}w(\cdot,t),\,w(\cdot,t)\right)_{L^2}\\&=\frac{1}{2}\frac{\partial}{\partial t}\|v(\cdot,t)\|_{L^2}^2+\cos((\beta/2-1)\pi)\|{}_aD_x^{\beta/2-1}w(\cdot,t)\|_{L^2}^2.\end{aligned} \tag{5.14}$$

We divide the right hand side of (5.13) into five parts

$$\mathcal{B}(\xi_u,-\xi_p,\xi_q;v,w,z)=I_1+I_2+I_3+I_4+I_5, \tag{5.15}$$

where

$$\begin{aligned}I_1&=\left(\frac{\partial\xi_u}{\partial t},\,v\right)_{L^2},\\I_2&=(\xi_q,\,w)_{L^2}+\left({}_aD_x^{\beta-2}\xi_p,\,w\right)_{L^2},\\I_3&=\sqrt{d}\left(\xi_q,\,\frac{\partial v}{\partial x}\right)_{L^2}+\sqrt{d}\left(\xi_u,\,\frac{\partial z}{\partial x}\right)_{L^2}-(\xi_p,\,z)_{L^2},\\I_4&=\sqrt{d}\sum_{j=1}^{M-1}\hat{\xi}_q[v]\big|_{x_j}+\sqrt{d}\sum_{j=1}^{M-1}\hat{\xi}_u[z]\big|_{x_j},\\I_5&=\sqrt{d}\left(\xi_q^+v^+\big|_a-\xi_q^-v^-\big|_b\right).\end{aligned} \tag{5.16}$$

Here the notation for jump of a piecewise function $b(x)$ at cell interface x_j is defined as

$$[b]\big|_{x_j}=b^+(x_j)-b^-(x_j). \tag{5.17}$$

Using the approximation results in Lemma 8, we obtain

$$I_1\le\frac{1}{2}\left\|\frac{\partial\xi_u}{\partial t}\right\|_{L^2}^2+\frac{1}{2}\|v\|_{L^2}^2\le ch^{2k+2}+\frac{1}{2}\|v\|_{L^2}^2, \tag{5.18}$$

where $c>0$ is a generic constant in this proof.

All the terms in I_3 vanish due to orthogonality.

For terms in I_4, when taking $\hat{\xi}_q=\xi_q^-$ and $\hat{\xi}_u=\xi_u^+$, we use $\xi_q=\mathcal{P}^-q-q$ and $\xi_u=\mathcal{P}^+u-u$; and when taking $\hat{\xi}_q=\xi_q^+$ and $\hat{\xi}_u=\xi_u^-$,we choose $\xi_q=\mathcal{P}^+q-q$ and $\xi_u=\mathcal{P}^-u-u$. Hence, both terms in I_4 vanish, and thus $I_4=0$.

For I_5, we have

$$I_5\le\frac{\sqrt{d}}{2}\left[\left(\xi_q^+(a,t)\right)^2+\left(v^+(a,t)\right)^2+\left(\xi_q^-(b,t)\right)^2+\left(v^-(b,t)\right)^2\right]. \tag{5.19}$$

Note that one of the two terms $\left(\xi_q^+(a,t)\right)^2$ and $\left(\xi_q^-(b,t)\right)^2$ vanishes, no matter which projection is used in the definition of ξ_q (5.12) is chosen. For example, taking $\xi_q=\mathcal{P}^+q-q$, and then $\left(\xi_q^+(a,t)\right)^2$ vanishes, and $\left(\xi_q^-(b,t)\right)^2\le Ch^{2k+2}$ by standard approximation of point values of ξ_q. Therefore,

$$I_5\le ch^{2k+2}+c\|v\|_{L^\infty}^2\le ch^{2k+2}+c\|v\|_{L^2}^2. \tag{5.20}$$

For terms in I_2, similarly as for terms of $\mathcal{V}$ in [7], using inequality $xy \leq \frac{x^2}{2\epsilon} + \frac{\epsilon y^2}{2}$, Lemma 2, Lemma 3, Lemma 4 and norm equivalence, we have

$$\begin{aligned} I_2 &\leq \frac{1}{2\epsilon}\|\xi_p\|_{L^2}^2 + \frac{1}{2\epsilon}\|\xi_q\|_{L^2}^2 + \frac{\epsilon}{2}\|w\|_{L^2}^2 + \frac{\epsilon}{2}\|{}_aD_x^{\beta-2}w\|_{L^2}^2 \\ &\leq \frac{c}{\epsilon}h^{2k+2} + c\epsilon\|{}_aD_x^{\frac{\beta}{2}-1}w(\cdot,t)\|_{L^2}^2, \end{aligned} \tag{5.21}$$

where ϵ is a small number that needs to be chosen so that the second term in (5.21) can be controlled by the second term in (5.14).

Combining the above estimations and integrating over time, we have

$$\begin{aligned} \frac{1}{2}\int_0^t \frac{\partial}{\partial s}\|v(\cdot,s)\|_{L^2}^2\,ds + (\cos(\beta/2-1)\pi - c\epsilon)\int_0^t \|{}_aD_x^{\frac{\beta}{2}-1}w(\cdot,s)\|_{L^2}^2\,ds \\ \leq (2c + c/\epsilon)h^{2k+2} + c\int_0^t \|v(\cdot,s)\|_{L^2}^2\,ds. \end{aligned} \tag{5.22}$$

We choose a sufficiently small ϵ, such that $c\epsilon < \cos(\beta/2-1)\pi$. Therefore, we have

$$\begin{aligned} &\frac{1}{2}\|v(\cdot,t)\|_{L^2}^2 + (\cos(\beta/2-1)\pi - c\epsilon)\int_0^t \|{}_aD_x^{\frac{\beta}{2}-1}w(\cdot,s)\|^2\,ds \\ &\qquad \leq \frac{1}{2}\|v(\cdot,0)\|_{L^2}^2 + ch^{2k+2} + c\int_0^t \|v(\cdot,s)\|_{L^2}^2 ds \\ &\qquad \leq ch^{2k+2} + c\int_0^t \|v(\cdot,s)\|_{L^2}^2\,ds \end{aligned} \tag{5.23}$$

By using the Grönwell lemma, we have

$$\|v(\cdot,t)\|_{L^2}^2 \leq ch^{2k+2}. \tag{5.24}$$

Together with the approximation result for ξ_u from Lemma 8, this result completes the proof. □

6 Numerical Tests

In this section, we present some numerical examples to validate the approximate theory and error estimates.

6.1 Numerical examples on function approximations

We first test on performance of standard L^2 projections of a function with weak singularity on uniform mesh.

Example 4 Consider three functions

$$\begin{aligned} w(x) &= x^{\frac{3}{2}}, \\ u(x) &= x^{\frac{4}{3}}, \\ v(x) &= e^{x^{\frac{3}{4}}(1-x)^{\frac{3}{4}}} - 1, \end{aligned} \tag{6.1}$$

Table 1 The L^2 errors of standard piecewise polynomial functions to approximate target functions w (6.1) on uniform mesh of $[0,1]$.

h	$k=1$		$k=2$		$k=3$	
	error	order	error	order	error	order
1/16	1.39e-03		1.52e-04		4.56e-05	
1/32	3.58e-04	1.96	3.82e-05	2.00	1.14e-05	2.00
1/64	9.22e-05	1.95	9.56e-06	2.00	2.85e-06	2.00
1/128	2.37e-05	1.95	2.39e-06	2.00	7.13e-07	2.00
1/256	6.09e-06	1.96	5.97e-07	2.00	1.78e-08	2.00

on $[0,1]$. We approximate the functions using piecewise polynomials with degree up to k ($k=1,2,3$) on uniform meshes. The errors measured in L^2 norm are shown in Table 1, Table 2 and Table 3. One can find the accuracy orders are 2 for w, 1.83 for u, and 1.25 for v, not $k+1$. This is due to the lack of regularity. The observation is summarized as follows: for a H^α function, the L^2 projection onto finite element space with piecewise polynomials with up to k degree achieves an error with numerical order $\min\{k+1, \alpha+0.5\}$, which is the same as described in Section 1.

Table 2 The L^2 errors of standard piecewise polynomial functions to approximate target functions u (6.1) on uniform mesh of $[0,1]$.

h	$k=1$		$k=2$		$k=3$	
	error	order	error	order	error	order
1/16	3.52e-04		4.99e-05		1.55e-05	
1/32	9.97e-05	1.82	1.40e-05	1.83	4.37e-06	1.83
1/64	2.82e-05	1.82	3.93e-06	1.83	1.22e-06	1.83
1/128	7.96e-06	1.82	1.10e-06	1.83	3.44e-07	1.83
1/256	2.24e-06	1.82	3.09e-07	1.83	9.65e-08	1.83

Table 3 The L^2 errors of standard piecewise polynomial functions to approximate target functions v (6.1) on uniform mesh of $[0,1]$.

h	$k=1$		$k=2$		$k=3$	
	error	order	error	order	error	order
1/64	4.77e-04		1.71e-04		8.72e-05	
1/128	2.01e-04	1.24	7.22e-05	1.24	3.67e-05	1.24
1/256	8.45e-05	1.24	3.04e-05	1.24	1.54e-05	1.25
1/512	3.56e-05	1.25	1.28e-05	1.25	6.50e-06	1.25
1/1028	1.45e-05	1.25	5.38e-06	1.25	2.73e-06	1.25

Next, we test on the performance of proposed space on the paired mesh.

Example 5 For function w in (6.1), we use $V_{1/2}^k$ on mesh in Example 1 to approximate. The L^2errors are listed in Table 4, and they are like $\mathcal{O}(h^{k+1})$ for any $k=1$, 2, and 3.

Example 6 For function u in (6.1), we use $V_{1/3}^k$ on mesh in Example 2 to approximate. The L^2errors are listed in Table 5, and they are like $\mathcal{O}(h^{k+1})$.

Table 4 The L^2 errors of $V_{1/2}^k$ to approximate target function w (6.1) on mesh in Example 1.

h	$k=1$		$k=2$		h	$k=3$	
	error	order	error	order		error	order
1/8	3.33e-03		1.69e-04		1/8	2.79e-06	
1/16	8.21e-04	2.02	1.98e-05	2.81	1/16	2.02e-07	3.78
1/32	2.03e-04	2.01	2.44e-06	2.91	1/24	4.15e-08	3.90
1/64	5.05e-05	2.00	3.05e-07	2.95	1/32	1.33e-08	3.94
1/128	1.26e-05	2.00	3.81e-08	2.98	1/40	5.49e-09	3.96

Table 5 The L^2 errors of $V_{1/3}^k$ to approximate target function u (6.1) on mesh in Example 2.

h	$k=1$		$k=2$		h	$k=3$	
	error	order	error	order		error	order
1/8	6.00e-04		2.31e-05		1/8	9.01e-07	
1/16	1.51e-04	1.98	2.99e-06	2.95	1/16	6.49e-08	3.79
1/32	3.78e-05	1.99	3.78e-07	2.99	1/24	1.33e-08	3.90
1/64	9.45e-06	2.00	4.73e-08	3.00	1/32	4.28e-09	3.94
1/128	2.36e-06	2.00	5.92e-09	3.00	1/40	1.82e-09	3.84

Example 7 Use space $V_{1/3}^k$ and mesh in Example 2 to approximate

$$v(x) = e^{x^{\frac{4}{3}}} - 1 \tag{6.2}$$

on $[0,1]$. The L^2 errors are $\mathcal{O}(h^{k+1})$, as listed in Table 6.

Table 6 The L^2 errors of $V_{1/3}^k$ to approximate function v (6.2) on mesh in Example 2.

h	$k=1$		$k=2$		h	$k=3$	
	error	order	error	order		error	order
1/8	7.74e-04	0.0	2.64e-05	0.0	1/8	2.02e-06	0.0
1/16	1.91e-04	2.01	3.61e-06	2.87	1/16	1.62e-07	3.64
1/32	4.74e-05	2.01	4.65e-07	2.96	1/24	3.43e-08	3.83
1/64	1.17e-05	2.01	5.86e-08	2.99	1/32	1.11e-08	3.90
1/128	2.93e-06	2.01	7.35e-09	3.00	1/40	4.63e-09	3.93

It is interesting to find out a mesh modified with graded mesh around the weak singularity and the proposed finite element space could recover the optimal order of accuracy in approximation. Out of curiosity, we test on the same space but a mesh modified with different graded mesh as in Example 1, which is not "graded" enough for the case.

Example 8 For function u in (6.1), using space $V_{1/3}^k$ on mesh defined in Example 1 gives errors shown in Table 7. The results show that a slightly graded mesh helps relieve the ill-conditioned trouble, but is less accurate compared with the paired mesh in Example 2.

Table 7 The errors of $V_{1/3}^k$ to approximate function u (6.1) on mesh defined in Example 1.

h	$k=1$		$k=2$		$k=3$	
	error	order	error	order	error	order
1/4	3.41e-03	0.0	3.57e-04	0.0	3.06e-05	0.0
1/8	9.36e-04	2.03	3.87e-05	3.20	2.43e-06	3.64
1/12	3.71e-04	2.00	1.09e-05	3.11	5.53e-07	3.65
1/16	2.08e-04	2.00	4.51e-06	3.07	1.93e-07	3.65
1/32	1.33e-04	2.00	2.29e-06	3.05	8.54e-08	3.65

Table 8 The error and order of convergence for LDG methods solving problem (1.2) with space V^1 on uniform mesh and $V_{1/2}^1$ on mesh defined in Example 2 at $T=1$.

h	V^1		h	$V_{1/2}^1$	
	error	order		error	order
1/8	7.13e-04		1/4	2.41e-03	
1/16	1.82e-04	1.97	1/8	6.17e-04	1.96
1/32	4.70e-05	1.95	1/16	1.41e-04	2.12
1/64	1.23e-05	1.92	1/32	3.46e-05	2.03
1/128	3.23e-06	1.93	1/64	8.73e-06	1.98

Table 9 The error and order of convergence for LDG methods solving problem (1.2) with space V^2 on uniform mesh and $V_{1/2}^2$ on mesh defined in Example 2 at $T=1$.

h	V^2		h	$V_{1/2}^2$	
	error	order		error	order
1/8	6.27e-05		1/4	9.85e-05	
1/16	1.48e-05	2.07	1/8	1.17e-05	3.06
1/32	3.65e-06	2.02	1/16	1.42e-06	3.03

6.2 Numerical examples for solving FDEs

In this section, we test on the performance of proposed scheme (3.3) with initialization (3.6), and Runge-Kutta methods (RK4) for time discretization for problem (1.1) with solution of form (2.9).

Example 9 Consider equation (1.2) on the computational domain $x \in \Omega = (0,1)$. Given initial condition

$$u_0(x) = x^{1.5}, \tag{6.3}$$

and Dirichlet boundary conditions

$$u(0,t) = 0, \quad u(1,t) = e^{-t}, \tag{6.4}$$

the exact solution is $u(x,t) = e^{-t}x^{1.5}$. We simulate the problem with (i) space V^k of standard piecewise polynomials with degree up to k on uniform mesh; and (ii) space $V_{1/2}^k$ of piecewise mapped polynomials with degree up to k on mesh in Example 1. The errors, measured in L^2 norm, are listed in Table 8 for $k=1$ and Table 9 for $k=2$. It is clear that by using a suitable approximation space, we can get optimal convergence rate $O(h^{k+1})$.

Example 10 Consider

$$\frac{\partial u(x,t)}{\partial t} = \frac{9\sqrt{3}\Gamma(\frac{2}{3})}{8\pi}\frac{\partial^{\frac{4}{3}}u(x,t)}{\partial x^{\frac{4}{3}}} - e^{-t}(x^{\frac{4}{3}}+1) \tag{6.5}$$

Table 10 The error and order of convergence for LDG methods solving problem (6.5) with space $V_{1/3}^k$ and mesh defined in Example 2, at $T = 0.01$.

h	$V_{1/3}^1$, cfl $= 0.00005$		$V_{1/3}^2$, cfl $= 0.00005$		$V_{1/3}^3$, cfl $= 0.0001$	
	error	order	error	order	error	order
1/8	9.76e-4		3.74e-5		1.41e-06	
1/16	2.92e-4	1.74	4.77e-6	2.97	1.04e-07	3.75
1/24	1.41e-5	1.78	1.40e-6	3.01	2.17e-08	3.86
1/32	8.55e-5	1.75	6.34e-7	2.75		
1/40	6.33e-5	1.34				

Table 11 The error and order of convergence for LDG methods solving problem (6.5) with V^1 on uniform mesh and $V_{1/3}^1$ on mesh defined in Example 1, at $T = 0.1$ with cfl $= 0.0001$.

h	V^1		h	$V_{1/3}^1$	
	error	order		error	order
1/8	1.44e-3		1/4	6.23e-03	
1/16	4.14e-4	1.80	1/8	1.76e-03	1.82
1/24	1.98e-4	1.81	1/12	7.47e-04	2.12
1/32	1.18e-4	1.79	1/16	4.22e-04	1.98
1/40	7.89e-5	1.81	1/20	2.64e-04	2.09
1/48	5.65e-5	1.82	1/24	1.85e-04	1.96

Table 12 The error and order of convergence for LDG methods solving problem (6.5) with V^2 on uniform mesh and $V_{1/3}^2$ on mesh defined in Example 1, at $T = 0.1$ with cfl $= 0.0001$.

h	V^2		h	$V_{1/3}^2$	
	error	order		error	order
1/8	2.10e-04		1/4	5.19e-04	
1/16	5.82e-05	1.85	1/8	5.55e-05	3.21
1/24	2.78e-05	1.82	1/12	1.58e-05	3.07
1/32	1.65e-05	1.81	1/16	7.25e-06	2.82
1/40	1.11e-05	1.74	1/20	2.15e-06	2.49

in $\Omega = (0, 1)$ with exact solution $u(x) = e^{-t}x^{\frac{4}{3}}$. We apply the proposed method with space $V_{1/3}^k$ on mesh in Example 2. Note that the time step τ is determined by the traditional mesh size, which is the length of smallest cell $I_1 = [x_0, x_1] = h^3$: $\tau =$ cfl $(h^{3k})^{1/4}$. Refining mesh size h will reduce the step size τ significantly. The results at $T = 0.01$ are in Table 10. The simulation for $k = 2$ is about 3rd order accurate.

For consideration of reducing computational cost, we would like to test on performance for novel basis with a slightly graded mesh defined in Example 1, which has smaller cell total number for similar mesh size. In this test, the time step is determined by $\tau =$ cfl $(h^{2k})^{1/4}$. The result for a longer time $T = 0.1$ is shown in Table 11 and 12. With sufficiently small time steps, the order of accuracy is optimal $k + 1$ before the mesh becomes not compatible for smaller h.

We also tried to use the mesh in Example 3, which allows bigger time steps $\tau =$ cfl $(\tilde{h}^{3k})^{1/4}$, and the result is shown in Table 13 - 14.

Table 13 The error and order of convergence for LDG methods with V^1 on uniform mesh and $V^1_{1/3}$ on mesh defined in Example 3, $T = 0.1$, cfl $= 0.01$.

h	V^1		$\{\tilde{h}, h\}$	$V^1_{1/3}$	
	error	order		error	order
1/8	1.44e-03		{1/8,1/32}	4.43e-04	
1/16	4.14e-04	1.80	{1/16,1/64}	2.29e-04	0.94
1/24	1.98e-04	1.81	{1/24,1/96}	1.22e-04	1.54
1/32	1.18e-04	1.79	{1/32,1/128}	7.41e-05	1.75
1/40	7.89e-05	1.81	{1/40,1/160}	4.90e-05	1.85
1/48	5.65e-05	1.82	{1/48,1/192}	3.62e-05	1.65

Table 14 The error and order of convergence for LDG methods with V^2 on uniform mesh and $V^2_{1/3}$ on mesh defined in Example 3, $T = 0.1$, cfl $= 0.01$.

h	V^2		$\{\tilde{h}, h\}$	$V^2_{1/3}$	
	error	order		error	order
1/8	2.10e-04		{1/8,1/32}	1.28e-03	
1/16	5.82e-05	1.85	{1/16,1/64}	2.38e-04	2.42
1/24	2.78e-05	1.82	{1/24,1/96}	6.83e-05	3.08
1/32	1.65e-05	1.81	{1/32,1/128}	2.90e-05	2.97
1/40	1.11e-05	1.74	{1/40,1/160}	1.47e-05	3.02

7 Conclusion

We design a novel LDG method for spatial fractional diffusion equations with weakly singular solutions and obtain optimal convergence rate. The key points are to build the singular information into finite element space, and modify the mesh with graded cells near the singularity to overcome the ill-condition mass matrix. Both stability and error estimation have been established. The global nature of the fractional operator makes computational cost really high, but the proposed approximation space gives optimal order of accuracy, and thus reduces the mesh size for same level of accuracy. However, the regularity of solution needs to be estimated ahead of time to choose a suitable approximation space. In this paper, we deal with the cases with fractional order, that is $\beta = \frac{p}{q}$ for some integers p and q. In the future, we plan to extend the work to more general cases. The methodology has potential to be applied to other fractional differential equations or any PDEs with such singular solutions, and this is also part of our future work.

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Conflict of interest

The authors declare that they have no conflict of interest.

A Proof of Lemma 6

Proof It is sufficient to show the approximation rates in the two subdomains $[a, \hat{x}]$ and $[\hat{x}, b]$ are optimal with order $k+1$. For simplicity, we assume $a = 0$ and $b = 1$.

In the domain $[\hat{x}, 1]$, the equation is considered to be smooth enough to satisfy the standard approximation theorem (Theorem 5), and

$$\|\mathcal{S}\omega - \omega\|^2_{L^2[\hat{x},1]} \leq Ch^{2k+2}. \tag{A.1}$$

Corresponding to the domain $[0, \hat{x}]$, we need to notice that function ω, considered as a function of $y = (x-a)^\gamma$, is a smooth function. Therefore, on any mapped cell $\tilde{I}_j$ $(j = 1, 2, \ldots, n)$, there exits a function $\tilde{\omega}_h \in P^k(x^\gamma)$, i.e., $\tilde{\omega}_h \circ x \in P^k(y)$, such that,

$$|\omega \circ x(y) - \tilde{\omega}_h \circ x(y)| \leq C\|\omega \circ x\|_{H^{k+1}(\tilde{I}_j)} h^{k+\frac{1}{2}}, \quad \forall y \in \tilde{I}_j. \tag{A.2}$$

Since $\mathcal{S}\omega$ is the least square approximation of ω in space V^k_γ,

$$\begin{aligned}
\|\mathcal{S}\omega - \omega\|^2_{L^2[I_j]} \leq \|\tilde{\omega}_h - \omega\|^2_{L^2[I_j]} &= \int_{x_{j-1}}^{x_j} |\omega(x) - \tilde{\omega}_h(x)|^2 \, dx \\
&= \int_{y_{j-1}}^{y_j} |\omega \circ x(y) - \tilde{\omega}_h \circ x(y)|^2 \frac{dx}{dy} \, dy \\
&= \int_{y_{j-1}}^{y_j} |\omega \circ x(y) - \tilde{\omega}_h \circ x(y)|^2 \frac{x^{1-\gamma}}{\gamma} \, dy \\
&\leq \frac{\hat{x}^{1-\gamma}}{\gamma} \int_{y_{j-1}}^{y_j} |\omega \circ x(y) - \tilde{\omega}_h \circ x(y)|^2 \, dy \\
&\leq C\|\omega \circ x\|^2_{H^{k+1}(\tilde{I}_j)} h^{2k+1} h.
\end{aligned} \tag{A.3}$$

Summing over all $j = 1, 2, \ldots, n$ gives

$$\|\mathcal{S}\omega - \omega\|^2_{L^2[0,\hat{x}]} \leq C\|\omega \circ x\|^2_{H^{k+1}([0,\hat{x}^\gamma])} h^{2k+2}. \tag{A.4}$$

Together with (A.1), the result in (A.4) gives

$$\|\mathcal{S}\omega - \omega\|^2_{L^2} = \|\mathcal{S}\omega - \omega\|^2_{L^2[0,\hat{x}]} + \|\mathcal{S}\omega - \omega\|^2_{L^2[\hat{x},1]} \leq Ch^{2k+2}. \tag{A.5}$$

□

B Proof of Lemma 7

Proof Similarly as in the proof of Theorem (6), it is sufficient to show the approximation rates in the two subdomains $[a, \hat{x}]$ and $[\hat{x}, b]$ are optimal with order $k+1$. Again for simplicity, we assume $a = 0$ and $b = 1$. And (A.1) holds true. One only needs to prove

$$\|\tilde{\mathcal{P}}_h\omega - \omega\|^2_{L^2[0,\hat{x}]} \leq Ch^{2k+2}. \tag{B.1}$$

Corresponding to domain $[0, \hat{x}]$, we need to notice that ω will be a smooth function with respect to $y = (x-a)^\gamma$.

$$\begin{aligned}
\|\tilde{\mathcal{P}}_h\omega - \omega\|^2_{L^2[0,\hat{x}]} &= \sum_{j=1}^n \int_{x_{j-1}}^{x_j} (\tilde{\mathcal{P}}_h\omega - \omega)^2 dx = \sum_{j=1}^n \int_{y_{j-1}}^{y_j} (\tilde{\mathcal{P}}_h\omega - \omega)^2 \frac{dx}{dy} dy \\
&= \sum_{j=1}^n \int_{y_{j-1}}^{y_j} (\tilde{\mathcal{P}}_h\omega - \omega)^2 \frac{x^{1-\gamma}}{\gamma} dy \leq \frac{\hat{x}^{1-\gamma}}{\gamma} \sum_{j=1}^n \int_{y_{j-1}}^{y_j} (\tilde{\mathcal{P}}_h\omega - \omega)^2 dy \leq Ch^{2k+2},
\end{aligned} \tag{B.2}$$

where in the last inequality, the standard approximation theorem in Lemma 5 is applied. This finishes the proof of (5.3). □

C Proof of Lemma 8

Proof Again we assume $a = 0$ and $b = 1$, and it is sufficient to only prove the results for subdomain $[0, \hat{x}]$. By the definition of $\mathcal{P}^-$ in (5.4), there exits a function $w_{k-1} \in V_\gamma^{k-1}$, such that

$$\mathcal{P}^- u = u^-(x_j) + (x^\gamma - x_j^\gamma) w_{k-1}. \tag{C.1}$$

Let $w = \frac{u-u^-(x_j)}{x^\gamma - x_j^\gamma}$, then w_{k-1} is a weighted L^2 projection of function w in V_γ^{k-1} with weight function $(x^\gamma - x_j^\gamma)$. Since $w \circ x(y)$ is also a smooth function on mapped cell $\tilde{I}_j$, there exits a function $\tilde{w}_{k-1} \in P^{k-1}(\tilde{I}_j)$ such that

$$|\omega \circ x(y) - \tilde{\omega}_{k-1} \circ x(y)| \le C \|\omega \circ x\|_{H^k(\tilde{I}_j)} h^{k-\frac{1}{2}}, \quad \forall y \in \tilde{I}_j. \tag{C.2}$$

Similarly, we have

$$\begin{aligned} \|\mathcal{P}^- u - u\|^2_{L^2[I_j]} &= \int_{x_{j-1}}^{x_j} (\mathcal{P}^- u - u)^2 \, dx = \int_{x_{j-1}}^{x_j} (x^\gamma - x_j^\gamma)^2 (w_{k-1}(x) - w(x))^2 \, dx \\ &\le \int_{x_{j-1}}^{x_j} (x^\gamma - x_j^\gamma)^2 (\tilde{\omega}_{k-1}(x) - w(x))^2 \, dx \\ &\le C h^2 \|\omega \circ x\|^2_{H^k(\tilde{I}_j)} h^{2k-1} h. \end{aligned} \tag{C.3}$$

Summing over all I_j with $j = 1, 2, \ldots, n$ completes the proof for $\mathcal{P}^-$. The proof for projection $\mathcal{P}^+$ is done in a similar way. □

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